Simulation of quantum computation with magic states via Jordan-Wigner transformations

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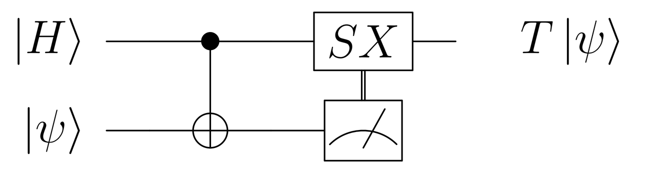
Abstract: Negativity in certain quasiprobability representations is a necessary condition for a quantum computational advantage. Here we define a new quasiprobability representation exhibiting this property with respect to quantum computations in the magic state model. It is based on generalized Jordan-Wigner transformations and it has a close connection to the probability representation of universal quantum computation based on the Λ polytopes. For each number of qubits it defines a polytope contained in the Λ polytope with some shared vertices. It leads to an efficient classical simulation algorithm for magic state quantum circuits for which the input state is positively represented, and it outperforms previous representations in terms of the states that can be positively represented.

Magic state quantum computation (QCM)

QCM [1] is a model of quantum computation in which:

- The allowed operations are restricted to stabilizer operations (Clifford gates and Pauli measurements). These operations alone are not universal for quantum computation and can be efficiently simulated classically.
- Universality is restored by additional nonstabilizer quantum states at the input of the circuit.

For example, a T gate is implemented by the following circuit



where $|H\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$.

Definitions

- Measurements in QCM are n-qubit Pauli observables which can be labelled by elements of $\mathbb{Z}_2^{2n}$. For any Pauli observable T_a, $a \in \mathbb{Z}_2^{2n}$, the projector corresponding to measurement outcome $s \in \mathbb{Z}_2$ is $\Pi_a^s := (1 + (-1)^s T_a)/2$.
- The Clifford group $\mathcal{C}\ell_n$ is the group of unitary gates that map Pauli operators to Pauli operators under conjugation.
- Let $\text{Herm}_1(\mathbb{C}^{2^n})$ denote the Hermitian operators on $\mathbb{C}^{2^n}$ with $\text{Tr} = 1$, and $\mathcal{S}_n$ the set of n-qubit stabilizer states.

Quasiprobability simulation methods

Define a set $\{A_\alpha \mid \alpha \in \mathcal{V}\} \subset \text{Herm}(\mathbb{C}^{2^n})$ with the properties:

1. $\text{span}(\{A_\alpha\}) = \text{Herm}(\mathbb{C}^{2^n})$, 2. $\text{Tr}(A_\alpha) = 1 \quad \forall \alpha$,
3. For any Clifford gate $g \in \mathcal{C}\ell$, $gA_\alpha g^\dagger = A_{g\cdot\alpha}$,
4. For any Pauli projector Π_a^s,
$$\Pi_a^s A_\alpha \Pi_a^s = \sum_\beta q_{\alpha,a}(\beta, s) A_\beta$$
with $q_{\alpha,a}(\beta,s) \geq 0$ and $\sum_{\beta,s} q_{\alpha,a}(\beta,s) = 1$.

Then in the quasiprobability representation:

- States are represented as
$$\rho = \sum_{\alpha\in\mathcal{V}} W_\rho(\alpha) A_\alpha$$
with $\sum_\alpha W_\rho(\alpha) = 1$ (follows from properties 1 & 2)
- Dynamics are represented by properties 3 & 4

Classical simulation of QCM

To simulate a circuit of the form:

$\rho_{in} \Rightarrow$ [Stabilizer operations: Clifford gates+ Pauli Measurements] $\Rightarrow$ Output

with $W_{\rho_{in}} \geq 0$,

1: sample $\alpha \in \mathcal{V}_n$ according to $p_\rho : \mathcal{V}_n \to \mathbb{R}_{\geq 0}$
2: propagate α through the circuit
3: **while** the end of the circuit has not been reached **do**
4: **if** a Clifford gate $g \in \mathcal{C}\ell_n$ is encountered **then**
5: update the phase space point according to $\alpha \rightarrowtail g\cdot\alpha$
6: **if** a Pauli measurement $a \in \mathbb{Z}_2^{2n}$ is encountered **then**
7: sample $(\beta, s) \in \mathcal{V}_n \times \mathbb{Z}_2$ according to $q_{\alpha,a}$
8: **return** $s \in \mathbb{Z}_2$ as the outcome of the measurement
9: update the phase space point according to $\alpha \rightarrowtail \beta$

This algorithm returns samples from the distribution of measurement outcomes for the quantum circuit being simulated which agree with the predictions of quantum theory.

Handling "negative probabilities"

- When $W_{\rho_{in}} \ngeq 0$, we simulate using the procedure from Ref. [2] by sampling from $P(\alpha) := W_{\rho_{in}}(\alpha)/||W_{\rho_{in}}||_1$.
- The cost of classical simulation (number of samples to achieve a given probability of error) scales with $||W_{\rho_{in}}||_1^2$.

The CNC model

Consider a set of Pauli observables $\Omega \subset \mathbb{Z}_2^{2n}$ such that

1. $\forall a, b \in \Omega$, $T_a T_b = T_b T_a \implies a + b \in \Omega$
2. $\exists \gamma : \Omega \to \mathbb{Z}_2$ such that $\forall a, b \in \Omega$
$$T_a T_b = T_b T_a \implies (-1)^{\gamma(a)+\gamma(b)} T_a T_b = (-1)^{\gamma(a+b)} T_{a+b}$$

The set of operators
$$\left\{ A_\Omega^\gamma := \frac{1}{2^n} \sum_{b\in\Omega} (-1)^{\gamma(b)} T_b \;\middle|\; \forall \Omega, \gamma \right\}$$
define a quasiprobability representation [3].

The CNC operators can be characterized:

Theorem 1 (Ref. 3) *For any CNC operator A_Ω^γ,*
$$A_\Omega^\gamma = g(A_{\tilde{\Omega}}^{\tilde{\gamma}} \otimes |\sigma\rangle\langle\sigma|)g^\dagger$$
where $g \in \mathcal{C}\ell$, $|\sigma\rangle$ is a stabilizer state, and
$$A_{\tilde{\Omega}}^{\tilde{\gamma}} = \frac{1}{2^n} \sum_{b\in\tilde{\Omega}} (-1)^{\tilde{\gamma}(b)} T_b, \text{ with } \{T_a, T_b\} = 2\delta_{a,b} \ \forall a, b \in \tilde{\Omega}.$$

T_bs have anti-commutation relations of Majorana operators.

Jordan-Wigner transformations

Any Hamiltonian can be expanded in Pauli operators
$$H = \sum_a c_a T_a.$$

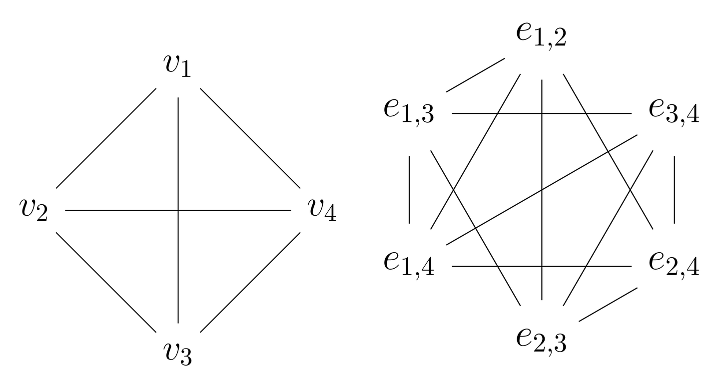
Definition 1 *The* frustration graph *$G(H)$ is a graph with*

- $a \in vert(G) \iff c_a \neq 0$
- $a \sim b \iff \{T_a, T_b\} = 0$

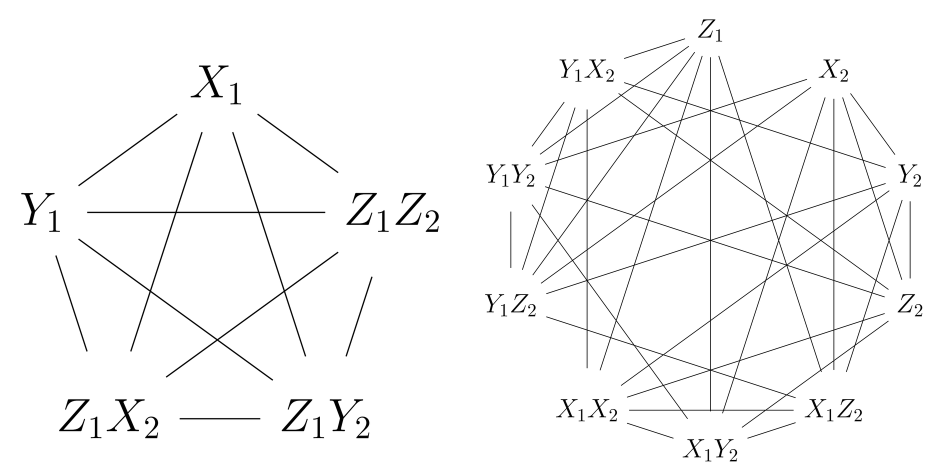
Theorem 2 (Ref. 4) *Given an n-qubit Hamiltonian in the Pauli basis for which the frustration graph G is the line graph of another graph R, then there exists a free-fermion description of H.*

Line graphs

$L(R)$ has a vertex for each edge of R; vertices in $L(R)$ are adjacent iff edges of R share a vertex.



R $\qquad$ $L(R)$



Quasiprobability model from JW transforms

We construct operators by the following procedure [5]

1. Choose a support $\Omega \subset \mathbb{Z}_2^{2n}$ such that $G(\Omega)$ is a line graph.
2. Project Λ onto span(Ω), the result is a new polytope Λ_Ω.
3. Choose a vertex $A \in \text{vert}(\Lambda_\Omega)$,
$$A = \sum_{a\in\Omega} c_a T_a$$
4. Choose a stabilizer state $|\sigma\rangle$ and a Clifford gate $g \in \mathcal{C}\ell$, return
$$g(A \otimes |\sigma\rangle\langle\sigma|)g^\dagger.$$

The set of operators constructed in this way defines a quasiprobability representation.

Connection to the Λ polytope model

For each number n of qubits, we define a polytope
$$\Lambda_n = \left\{X \in \text{Herm}\left(\mathbb{C}^{2^n}\right) \mid \text{Tr}(|\sigma\rangle\langle\sigma| X) \geq 0 \ \forall |\sigma\rangle \in \mathcal{S}_n\right\}.$$
Denote by $\{A_\alpha \mid \alpha \in \mathcal{V}_n\}$ the (finite) set of vertices of Λ_n.

Theorem 3 (Ref. 6) *For any number of qubits n,*

1. Any n-qubit quantum state ρ can be decomposed as
$$\rho = \sum_{\alpha\in\mathcal{V}_n} p_\rho(\alpha) A_\alpha,$$
with $p_\rho(\alpha) \geq 0$ for all $\alpha \in \mathcal{V}_n$, and $\sum_\alpha p_\rho(\alpha) = 1$.

2. For any A_α, $\alpha \in \mathcal{V}_n$, and any Clifford gate $g \in \mathcal{C}\ell_n$, $gA_\alpha g^\dagger$ is a vertex of Λ_n. This defines an action of the Clifford group on $\mathcal{V}_n$ as $gA_\alpha g^\dagger =: A_{g\cdot\alpha}$ where $g\cdot\alpha \in \mathcal{V}_n$.

3. For any A_α, $\alpha \in \mathcal{V}_n$, and any Pauli projector Π_a^s,
$$\Pi_a^s A_\alpha \Pi_a^s = \sum_{\beta\in\mathcal{V}_n} q_{\alpha,a}(\beta, s) A_\beta,$$
with $q_{\alpha,a}(\beta,s) \geq 0 \ \forall \beta, s$, and $\sum_{\beta,s} q_{\alpha,a}(\beta,s) = 1$.

This theorem describes a hidden variable model (HVM) for QCM in which (1) states are represented by probability distributions p_ρ over $\mathcal{V}_n$, (2) Clifford gates and Pauli measurements are represented by stochastic maps $g\cdot\alpha$ and $q_{\alpha,a}$.

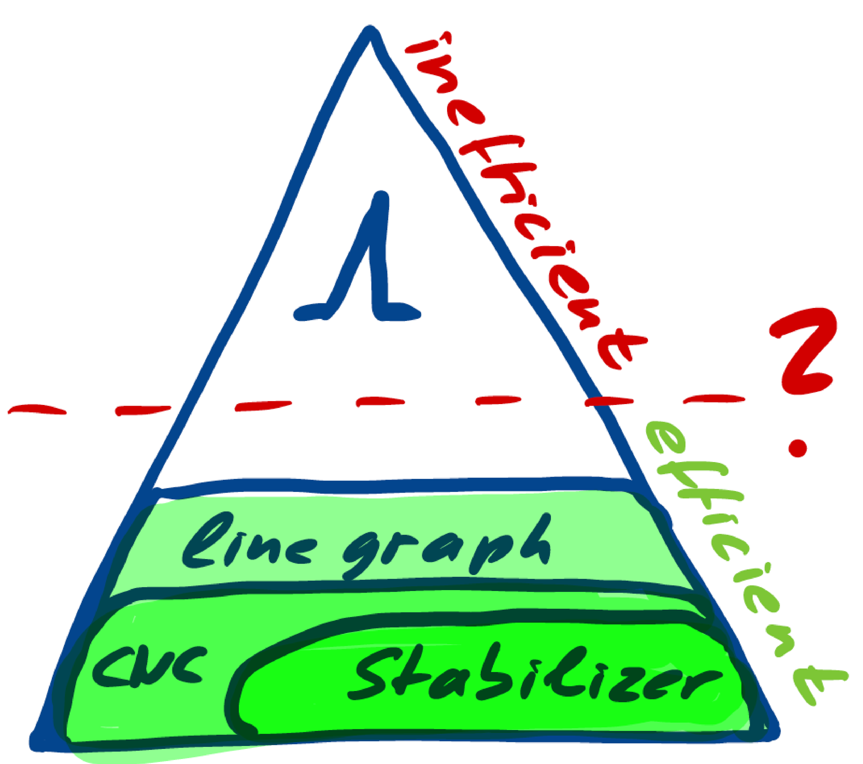
New vertices of Λ

Theorem 4 (Ref. 6) *For any CNC pair (Ω, γ), if Ω is maximal then*
$$A_\Omega^\gamma := \frac{1}{2^n} \sum_{a\in\Omega} (-1)^{\gamma(a)} T_a$$
is a vertex of Λ.

Theorem 5 (Ref. 5) *$\forall \Omega \subset \mathbb{Z}_2^{2n}$ s.t. $G(\Omega) \simeq L(K_{2n+1})$, there exists a choice of signs η such that*
$$A_\Omega^\eta = \frac{1}{2^n}\left(1 + \frac{1}{n}\sum_{b\in\Omega}(-1)^{\eta(b)} T_b\right)$$
is a vertex of Λ.

Main takeaway

The big question about the Λ polytopes [6] is where the line between efficient and inefficient classical simulation of quantum computation lies. Here, we increase the size of the known efficiently simulable region. Our construction is based on Jordan-Wigner transformations, and its range of applicability includes the earlier CNC construction [3], which in turn includes the stabilizer formalism.



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