The role of cohomology in quantum computation with magic states

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Abstract: A web of cohomological facts relates quantum error correction, measurement-based quantum computation, symmetry protected topological order and contextuality (see figure below). Here we extend this web to quantum computation with magic states [5]. In this computational scheme, the negativity of certain quasiprobability functions is an indicator for quantumness [3,4]. However, when constructing quasiprobability functions to which this statement applies, a marked difference arises between the cases of even and odd local Hilbert space dimension. At a technical level, establishing negativity as an indicator of quantumness in quantum computation with magic states relies on two properties of the Wigner function: their covariance with respect to the Clifford group and positive representation of Pauli measurements. In odd dimension, Gross' Wigner function—an adaptation of the original Wigner function to odd-finite-dimensional Hilbert spaces—possesses these properties. In even dimension, Gross' Wigner function doesn't exist. Here we discuss the broader class of Wigner functions that, like Gross', are obtained from operator bases. We find that such Clifford-covariant Wigner functions do not exist in any even dimension, and furthermore, Pauli measurements cannot be positively represented by them in any even dimension whenever the number of qudits is $n \geq 2$. We establish that the obstructions to the existence of such Wigner functions are cohomological.

Magic state quantum computation (QCM)

QCM is a universal model of quantum computation in which:

(i) The allowed operations are restricted to stabilizer operations (Clifford gates and Pauli measurements). These operations alone are not universal for quantum computation and can be efficiently simulated classically [1].

(ii) Universality is restored by additional nonstabilizer quantum states at the input of the circuit [2]. Thus, the computational power resides with the magic states.

Definitions

For qudits of dimension d, the Pauli operators are

$$X = \sum_{j\in\mathbb{Z}_d} |j+1 \bmod d\rangle\langle j|, \quad Z = \sum_{j\in\mathbb{Z}_d} \omega^j |j\rangle\langle j|$$

where $\omega = e^{2\pi i/d}$. Up to overall phases, the n-qudit Pauli operators are

$$T_a = \mu^{\gamma(a)} \bigotimes_{k=1}^{n} Z^{a_z[k]} X^{a_x[k]}, \ \forall a = (a_z, a_x) \in \mathbb{Z}_d^{2n} \qquad (1)$$

where $\mu = \omega$ ($\mu = \sqrt{\omega}$) if d is odd (even). The symplectic form on $\mathbb{Z}_d^{2n}$ tracks the commutator of the Pauli operators as $[T_a, T_b] := T_a T_b T_a^{-1} T_b^{-1} = \omega^{[a,b]}$, $\forall a, b \in \mathbb{Z}_d^{2n}$. We define a function β that tracks how commuting Pauli operators compose through

$$T_a T_b = \omega^{\beta(a,b)} T_{a+b}, \quad \forall a, b \in \mathbb{Z}_d^{2n} \text{ s.t. } [a,b] = 0. \qquad (2)$$

The Clifford group is the normalizer of the Pauli group in the unitary group: $\mathcal{C}\ell = \mathcal{N}(\mathcal{P})/U(1)$. The Clifford group acts on the Pauli operators as

$$g(T_a) = \omega^{\tilde{\Phi}_g(a)} T_{S_g a}, \quad \forall g \in \mathcal{C}\ell, \forall a \in \mathbb{Z}_d^{2n} \qquad (3)$$

where S_g is a symplectic map on $\mathbb{Z}_d^{2n}$.

Properties of discrete Wigner functions

We look for Wigner functions satisfying the properties:

OB (Operator basis): $\forall Y$ a Wigner function W_Y satisfies

$$Y = \sum_{v\in V} W_Y(v) A_v,$$

where $\{A_v\}_{v\in V}$ form an operator basis with phase space $V = \mathbb{Z}_d^n \times \mathbb{Z}_d^n$.

SW1 (Reality): $W_{Y^\dagger}(u) = (W_Y(u))^*$, $\forall u \in V$.

SW2 (Standardization): $\sum_{u\in V} W_Y(u) = \mathrm{Tr}(Y)$.

SW3 (Pauli covariance): $W_{T_a(Y)}(u+a) = W_Y(u)$, $\forall u, a$.

SW4 (Traciality): $\sum_{u\in V} W_{Y_1}(u)\Theta_{Y_2}(u) = \mathrm{Tr}(Y_1 Y_2)$.

Wigner functions for QCM

Wigner functions that are useful for describing QCM also satisfy the following properties:

Clifford covariance: A Wigner function is Clifford covariant if

$$W_{g(Y)}(S_g u + a_g) = W_Y(u) \quad \forall g \in \mathcal{C}\ell$$

where S_g is a symplectic map on $\mathbb{Z}_d^{2n}$.

Positive representation of Pauli measurement: A Wigner function positively represents Pauli measurement if

(i) For any Pauli projector $\Pi_{a,s}$, the Born rule takes the form

$$\mathrm{Tr}(\Pi_{a,s}\rho) = \sum_{v\in V} \Theta_{\Pi_{a,s}}(v) W_\rho(v)$$

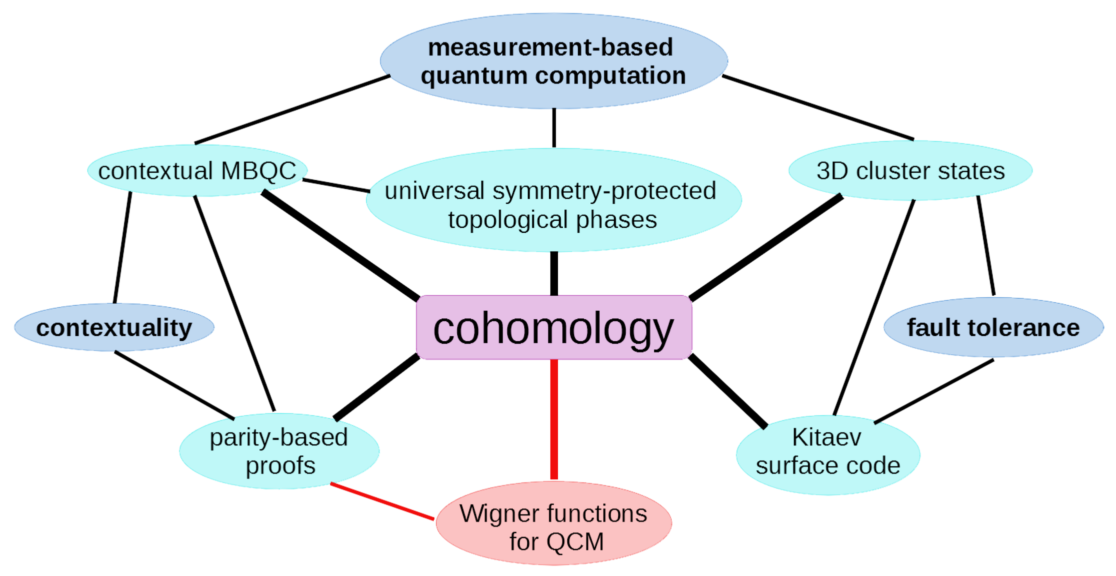
with $0 \leq \Theta_{\Pi_{a,s}}(v) \leq 1$, $\forall v, a, s$

(ii) For all Pauli projectors $\Pi_{a,s}$,

$$W_\rho \geq 0 \Longrightarrow W_{\Pi_{a,s}\rho\Pi_{a,s}} \geq 0$$

Our goal is to determine when a Wigner function satisfying (OB),(SW1)–(SW4) exists which is also Clifford covariant and positively represents Pauli measurement.

Role of cohomology in quantum computation



Cohomology

Let $\mathcal{C}_* = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ denote the chain complex where $\mathcal{C}_k$ is the free $\mathbb{Z}_d$-module with basis $[v_1|v_2|\cdots|v_k]$ such that $v_j \in E$ and $[v_i, v_j] = 0$. The boundary map ∂ is given by

$$\partial[v_1|v_2|\cdots|v_k] = [v_2|\cdots|v_k] + (-1)^k[v_1|\cdots|v_{k-1}] + \left(\sum_{i=1}^{k-1}(-1)^i[v_1|\cdots|v_i + v_{i+1}|\cdots|v_k]\right)$$

$\mathcal{C}^*$ is the cochain complex with k-cochains C^k $\mathbb{Z}_d$-module maps $f: \mathcal{C}_k \to \mathbb{Z}_d$, with coboundary map $df(-) = f(\partial -)$.

β is a 2-cochain. It follows from associativity $(T_a T_b)T_c = T_a(T_b T_c)$ that $d\beta(a,b) := \beta(b,c) - \beta(a+b,c) + \beta(a,b+c) - \beta(a,b) = 0$. Therefore β is a 2-cocycle.

For describing the Clifford group we define a (co)chain complex $\tilde{\mathcal{C}}$ as before except without the commutativity constraint. $\mathcal{C}^p(\mathcal{C}\ell, \tilde{\mathcal{C}}^q)$ is a bicomplex with two coboundaries:

$$\begin{array}{ccc} C^p(\mathcal{C}\ell, \tilde{C}^q) & \xrightarrow{d^h} & C^{p+1}(\mathcal{C}\ell, \tilde{C}^q) \\ d^v \downarrow & & \\ C^p(\mathcal{C}\ell, \tilde{C}^{q+1}) & & \end{array}$$

- d^v is induced by the boundary map ∂ on $\tilde{C}$
- d^h is a coboundary in group cohomology

$\tilde{\Phi}$ is a 1-cochain in $\mathcal{C}^p(\mathcal{C}\ell, \tilde{\mathcal{C}}^q)$. Again from associativity, $(gh)(T_a) = g(h(T_a))$, we have $(d^h\Phi)_{g,h} := \tilde{\Phi}_h(a) - \tilde{\Phi}_{gh}(a) + \tilde{\Phi}_g(S_h a)$ and so $\tilde{\Phi}$ is a 1-cocycle.

Motivation for cohomological arguments

Before defining a Wigner function an arbitrary phase function γ must be chosen for the Pauli operators in Eq. (1). When the phase convention is changed by

$$\gamma(a) \longrightarrow \begin{cases} \gamma(a) + \nu(a) & \text{if } d \text{ is odd} \\ \gamma(a) + 2\nu(a) & \text{if } d \text{ is even,} \end{cases}$$

the resulting change in β (see Eq. (2)) is

$$\begin{aligned} \beta(a,b) \longrightarrow & \beta(a,b) + \nu(a) + \nu(b) - \nu(a+b) \\ = & \beta(a,b) + d\nu(a,b). \end{aligned}$$

where $d\nu$ is the coboundary of ν. The equivalence class $[\beta] := \{\beta + d\nu \mid \nu \in \mathcal{C}^1\}$ is an element of a cohomology group $H^2(\mathcal{C}, \mathbb{Z}_d)$

The resulting change in $\tilde{\Phi}_g$ (see Eq. (3)) is

$$\begin{aligned} \tilde{\Phi}_g(a) \longrightarrow & \tilde{\Phi}_g(a) + \nu(a) - \nu(S_g a) \\ = & \tilde{\Phi}_g(a) - (d^h\nu)_g(a) \end{aligned}$$

where $d^h\nu$ is a coboundary in group cohomology. The equivalence class $[\tilde{\Phi}] := \{\tilde{\Phi} + d^h\nu \mid \nu \in \tilde{\mathcal{C}}^1\}$ is an element of the cohomology group $H^1(\mathcal{C}\ell, \tilde{\mathcal{C}}^1)$.

Physical properties should not depend on the phase convention so properties should depend on β and Φ only through their cohomology classes. It turns out, existence of a Clifford covariant Wigner function hinges on $[\tilde{\Phi}]$ (see Theorem 1) and existence of a Wigner function that positively represents Pauli measurement hinges on $[\beta]$ (see Theorem 2).

Clifford covariance

Let $\tilde{B}_1$ denote the image of the boundary map $\partial: \tilde{C}_2 \to \tilde{C}_1$, and U_{cov} the set of $\mathbb{Z}_d$-module maps $\tilde{B}_1 \to \mathbb{Z}_d$. We choose a set-theoretic section $\theta: Q \to \mathcal{C}\ell$ of the quotient map $\Pi: \mathcal{C}\ell \to Q$ where $Q = \mathcal{C}\ell/\mathcal{P}$ and $\mathcal{P} \subset \mathcal{C}\ell$ is the Pauli group. Then $\Phi_{cov} \in C^1(Q, U_{cov})$ is defined to be the composite

$$\Phi_{cov}: Q \xrightarrow{\theta} \mathcal{C}\ell \xrightarrow{\tilde{\Phi}} C^1 \xrightarrow{d^v} U_{cov}.$$

Then we have the following result:

Theorem 1 *For any number n of qudits of any dimension d, a Clifford covariant Wigner function satisfying (OB) exists if and only if $[\Phi_{cov}] = 0 \in H^1(Q, U_{cov})$.*

Implications for odd-dimensional qudits

Lemma 1 *For odd-dimensional qudits, $[\Phi_{cov}] = 0$.*

Corollary 1 *For any number of qudits n of any odd dimension d, a Clifford covariant Wigner function satisfying (OB) exists.*

Implications for even-dimensional qudits

Lemma 2 *For even-dimensional qudits, $[\Phi_{cov}] \neq 0$.*

Corollary 2 *For any number of qudits n of any even dimension d, a Clifford covariant Wigner function satisfying (OB) does not exists.*

See Ref. [5] for proofs.

Positive representation of Pauli measurement

Theorem 2 *For any number n of qudits of any dimension d, a Wigner function satisfying (OB), (SW1)–(SW4) that positively represents Pauli measurements exists if and only if $[\beta] = 0 \in H^2(\mathcal{C}, \mathbb{Z}_d)$.*

Odd-dimensional qudits

Lemma 3 *For odd-dimensional qudits, $[\beta] = 0$.*

Proof. Choose $\gamma = -\langle a_z | a_x \rangle \cdot 2^{-1}$

Corollary 3 *For any number of qudits of any odd dimension, a Wigner function that positively represents Pauli measurements exists.*

Even-dimensional qudits

Lemma 4 *For even-dimensional qudits, $[\beta] \neq 0$.*

Proof. A generalization of the Mermin square. Define $\tilde{X} = X^{d/2}$ and $\tilde{Y} = \sqrt{\omega}^{d/2} X^{d/2} Z$. Then the generalized Mermin square is

$T_a = Z^{-1} \otimes I$	$\times$	$T_b = I \otimes Z$	$\times$	$T_{a+b}^{-1} = Z \otimes Z^{-1}$	$= I \Rightarrow \beta(a,b) = 0$
$\times$		$\times$		$\times$	
$T_c = I \otimes \tilde{X}$	$\times$	$T_d = \tilde{X}^{-1} \otimes I$	$\times$	$T_{c+d}^{-1} = \tilde{X} \otimes \tilde{X}^{-1}$	$= I \Rightarrow \beta(c,d) = 0$
$\times$		$\times$		$\times$	
$T_{a+c}^{-1} = Z \otimes \tilde{X}^{-1}$	$\times$	$T_{b+d}^{-1} = \tilde{X} \otimes Z^{-1}$	$\times$	$T_{a+b+c+d} = \tilde{Y}^{-1} \otimes \tilde{Y}$	$= I \Rightarrow \beta(a+c, b+d) = 0$
$\Vert$		$\Vert$		$\Vert$	
I		I		$-I$	
$\Downarrow$		$\Downarrow$		$\Downarrow$	
$\beta(a,c) = 0$		$\beta(b,d) = 0$		$\beta(a+b, c+d) = d/2$	

Corollary 4 *For any number of qudits of any even dimension, a Wigner function that positively represents Pauli measurements does not exist.*

See Ref. [5] for proofs.

References

[1] D Gottesman. arXiv:quant-ph/9807006 (1998).

[2] S Bravyi, A Kitaev. Phys Rev A (2005).

[3] D Gross. PhD Thesis, Imperial College London (2008).

[4] V Veitch, C Ferrie, D Gross, J Emerson. New J Phys (2012).

[5] R Raussendorf, C Okay, M Zurel, P Feldmann. arXiv:2110.11631 (2021).