Hidden Variable Model for Quantum Computation with Magic States on Qudits of Any Dimension

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Abstract: It was recently shown that a hidden variable model can be constructed for universal quantum computation with magic states on qubits. Here we show that this result can be extended, and a hidden variable model can be defined for quantum computation with magic states on qudits with any Hilbert space dimension. This model leads to a classical simulation algorithm for universal quantum computation. Details are available at arXiv:2110.12318 [6].

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Introduction

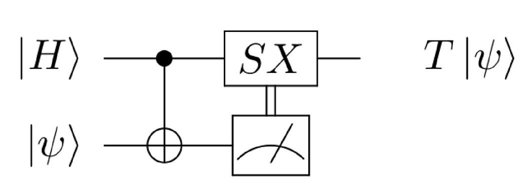
A key question at the heart of quantum computation remains without a satisfying answer: *what is the essential quantum resource that provides the computational speedup of quantum computation over classical computation?* The is the question we consider here.

One fruitful way to approach this question comes from quantum computation with magic states [1].

Quantum computation with magic states (QCM)

QCM is a universal model of quantum computation in which:

- The allowed operations are restricted to Clifford gates and Pauli measurements. These operations are not universal for quantum computation and can be efficiently simulated classically.
- Universality is restored by additional nonstabilizer quantum states at the input of the circuit. Thus, the computational power resides with the magic states, e.g. the $|H\rangle$ state below.



This model allows us to refine the question posed above. Instead of asking broadly *which nonclassical resources are required for a quantum computational speedup?*, we can focus on the quantum states and ask *which states could provide a speedup in QCM?*

A partial answer to this question is provided by the study of quasiprobability representations like the Wigner function.

Quasiprobability representations

Veitch et al. [2] showed that a necessary condition for a quantum computational speedup in QCM on odd-prime-dimensional qudits is that the discrete Wigner function of the input state of the quantum circuit must take negative values. The amount of negativity in the discrete Wigner function quantifies the cost of classical simulation of a quantum computation with simulation being efficient if the Wigner function is nonnegative everywhere [3]. Since nonnegativity of the discrete Wigner function also implies the existence of a classical (noncontextual) hidden variable model (HVM) describing the computation, this proves that two traditional notions of nonclassicality for quantum systems—Wigner negativity and failure of a classical HVM description—herald a quantum computational advantage over classical computation.

Similar necessary conditions for quantum advantage in QCM have been proven based on other quasiprobability representations. In all cases, negativity is required in the representation of states or operations in order to describe universal quantum computation.

Our contributions

Recently a hidden variable model was defined which represents all quantum states, operations, and measurements relevant for QCM on qubits using only classical (nonnegative) probabilities [5]. This model is structurally similar to previously defined quasiprobability representations for quantum computation and leads to a classical simulation method for universal quantum computation based on sampling.

We show that this result can be significantly extended in that a nonnegative hidden variable model can be constructed for quantum computation with magic states on any number of qudits of any dimension. This model also spawns a classical simulation algorithm for quantum computation with magic states.

Significance of results

- In previously defined quasiprobability representations for QCM, negativity was shown to be a necessary condition for a quantum computational speedup by defining an efficient classical simulation algorithm for the quantum computation that applied whenever the input state had a nonnegative representation.
- In our model, every state has a nonnegative probability representation so the classical simulation algorithm applies to QCM on any input state (including magic states).
- This classical simulation algorithm is not efficient in general. There are however some important cases in which it is efficient. E.g. the phase space point operators of Ref. [4] and the phase point operators of the odd-dimensional discrete Wigner function are special cases of the hidden variables of our model. When the support of the probability representation of the input state is restricted to these variables, the simulation algorithm is efficient.
- Characterizing more of the hidden variables of the model could expand the scope of efficient simulability of QCM.
- We also believe that the Λ polytopes could be of independent interest potentially with other applications.

Definitions

- Denote the n-qudit Pauli operators by
$$T_a = e^{i\phi(a)} \bigotimes_{k=1}^{n} Z^{a_z[k]} X^{a_x[k]}, \quad \forall a = (a_z, a_x) \in \mathbb{Z}_d^n \times \mathbb{Z}_d^n =: E,$$
where the local Pauli operators are
$$X = \sum_{j\in\mathbb{Z}_d} |j+1\rangle\langle j|, \quad Z = \sum_{j\in\mathbb{Z}_d} \omega^j |j\rangle\langle j|$$
with $\omega = \exp(2\pi i/d)$.
- For any set of pair-wise commuting Pauli observables $J \subset E$, denote the projector onto the eigenspace of the observables in J corresponding to eigenvalues $\{\omega^{r(a)} \mid a \in J\}$ by
$$\Pi_J^r := \frac{1}{|J|}\sum_{a\in J} \omega^{-r(a)} T_a.$$
- The Clifford group, denoted $\mathcal{C}\ell$, is the normalizer of the Pauli operators in the unitary group up to overall phases.
- Denote by $\mathrm{Herm}_1(d^n)$ the set of Hermitian, unit trace operators on n-qudit Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$.
- Let $\mathcal{S}$ denote the set of projectors onto pure n-qudit stabilizer states.

Main result

Define
$$\Lambda := \{X \in \mathrm{Herm}_1(d^n) \mid \mathrm{Tr}(\Pi X) \geq 0 \; \forall \Pi \in \mathcal{S}\}.$$

Λ is a closed, bounded polytope in $\mathrm{Herm}_1(d^n)$ with a finite number of vertices. Denote the vertices of Λ by $\{A_\alpha \mid \alpha \in \mathcal{V}\}$. We have the following theorem:

Theorem. For any number of qudits n with any dimension d, we have a finite set of Hermitian operators $\{A_\alpha \mid \alpha \in \mathcal{V}\}$ such that

(i) Each n-qudit quantum state ρ can be represented by a probability function $p_\rho : \mathcal{V} \longrightarrow \mathbb{R}_{\geq 0}$,
$$\rho = \sum_{\alpha\in\mathcal{V}} p_\rho(\alpha)\, A_\alpha. \tag{1}$$

(ii) For any $\alpha \in \mathcal{V}$ and any Clifford gate $g \in \mathcal{C}\ell$, it holds that
$$g(A_\alpha) = A_{g\cdot\alpha} \;\text{ with }\; g\cdot\alpha \in \mathcal{V}. \tag{2}$$

(iii) For any $\alpha \in \mathcal{V}$ and any Pauli measurement projector Π_J^r,
$$\Pi_J^r A_\alpha \Pi_J^r = \sum_{\beta\in\mathcal{V}} q_{\alpha,J}(\beta, r)\, A_\beta, \tag{3}$$
where for all α, J, the $q_{\alpha,J}$ are probability functions.

(iv) The probability of obtaining outcomes $r : J \longrightarrow \mathbb{Z}_d$ for measurements $J \subset E$ on the state ρ is given by
$$\mathrm{Tr}\left(\Pi_J^r \rho\right) = \sum_{\alpha\in\mathcal{V}} p_\rho(\alpha) Q_J(r|\alpha), \tag{4}$$
where $Q_J(r|\alpha)$ is given by
$$Q_J(r|\alpha) := \sum_{\beta\in\mathcal{V}} q_{\alpha,J}(\beta, r). \tag{5}$$
For any α and J, $Q_J(r|\alpha)$ is a probability distribution over the set of value assignments for J.

This theorem defines a hidden variable model that represents all components of quantum computation with magic states—Clifford gates (ii), Pauli measurements (iii-iv), and magic states (i)—by a family of probability distributions defined in eqs. (1)–(5).

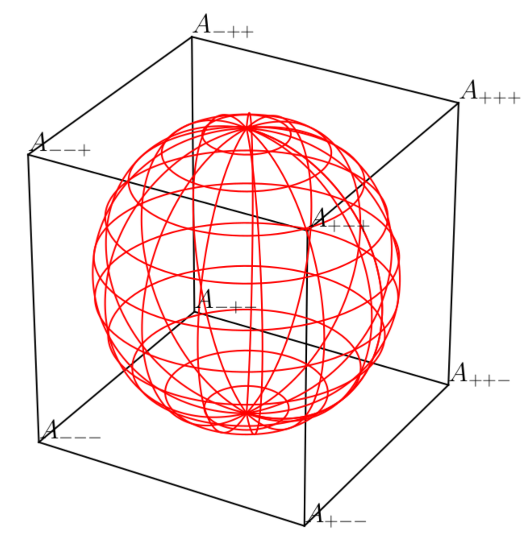
Classical simulation of universal quantum computation

The hidden variable model defined above can be used to simulate quantum computation with magic states.

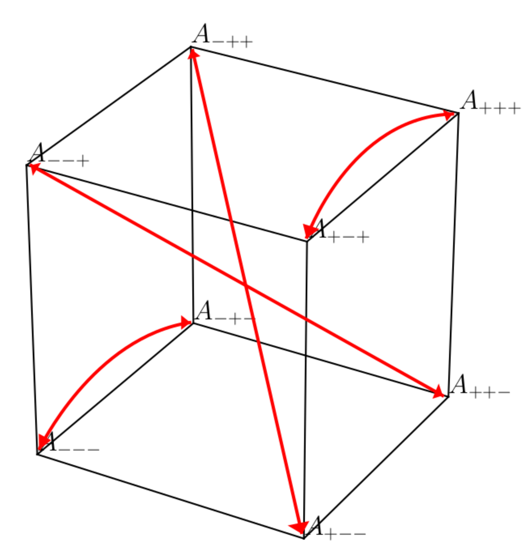
Algorithm sketch.

1. Start with a decomposition of the input state ρ of the form Eq. (1). Sample from $p_\rho(\alpha)$ to obtain an initial phase space point α.
2. For each Clifford gate g in the quantum circuit, update the phase space point $\alpha \longleftarrow g\cdot\alpha$.
3. For each Pauli measurement $a \in E$, sample from the distribution $q_{\alpha,\langle a\rangle}$ to obtain $r(a) \in \mathbb{Z}_d$ and $\beta \in \mathcal{V}$. Return $r(a)$ as the outcome of the measurement and update the phase space point $\alpha \longleftarrow \beta$.

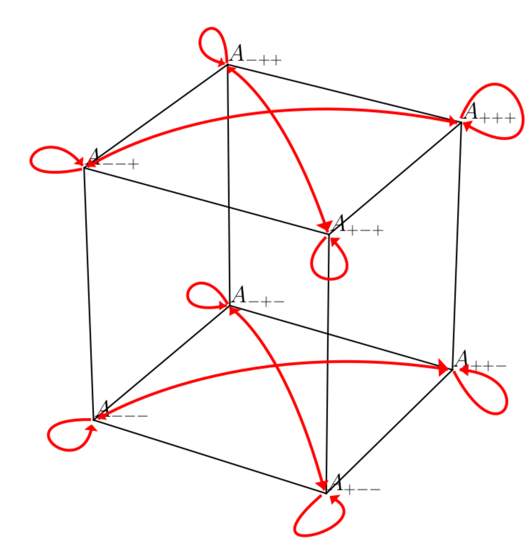
One qubit example



The state space Λ is a cube with eight vertices corresponding to the phase point operators $A_{\pm\pm\pm} = \frac{1}{2}[I \pm X \pm Y \pm Z]$. The physical one-qubit states lie on or in the Bloch sphere which is inscribed in Λ and touches the boundary of Λ at the 6 stabilizer states. Any one-qubit state can be expressed as a convex combination of the vertices of the cube as in Eq. (1).



Under a Clifford gate, vertices of Λ map deterministically to other vertices. For example, the action of a Hadamard gate H on the vertices is shown here. Each red arrow represents a deterministic transition.



Under a Pauli measurement, $a \in E$, outcome $r(a)$ is returned with probability $Q_a(r|\alpha)$ and the update of the vertices is probabilistic. For example, a Pauli Z measurement is shown here. Each red arrow represents a transition probability of 0.5.

Structure of the polytope Λ

The polytope Λ can only be easily visualized in the simplest case of one qubit above, but for any n, d the result is similar. In general,

- The dimension of the ambient space is $d^{2n} - 1$.
- All states are contained in Λ, so they can be described as a convex combination of the vertices of Λ as in eq. (1).
- Update under Clifford gates is deterministic, eq. (2).
- Update under Pauli measurements is probabilistic, eqs. (3)–(5).

Some facts are known about the polytopes, e.g.:

- For qubits, the phase point operators of the form
$$A_\Omega^\gamma = \frac{1}{2^n}\sum_{b\in\Omega} \omega^{-\gamma(b)} T_b$$
defined in Ref. [4] are vertices of Λ. For odd-dimensional qudits, the phase space point operators of the discrete Wigner function of the form
$$A_u = \frac{1}{d^n}\sum_{v\in E} \omega^{[u,v]} T_v$$
are vertices of Λ.
- For qubits, for any vertex A_α, any projector onto a stabilizer state Π, and any Clifford gate g,
$$g(A_\alpha \otimes \Pi)$$
is a vertex of Λ. For qudits, $g(A_\alpha \otimes \Pi)$ is in Λ but is not necessarily a vertex.

References

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