Phase-space-simulation method for quantum computation with magic states on qubits

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Abstract: We propose a method for classical simulation of finite-dimensional quantum systems, based on sampling from a quasiprobability distribution, i.e., a generalized Wigner function. Our construction applies to all finite dimensions, with the most interesting case being that of qubits. For multiple qubits, we find that quantum computation by Clifford gates and Pauli measurements on magic states can be efficiently classically simulated if the quasiprobability distribution of the magic states is non-negative. This provides the so far missing qubit counterpart of the corresponding result [V. Veitch et al., New J. Phys. 14, 113011 (2012)] applying only to odd dimension. Our approach is more general than previous ones based on mixtures of stabilizer states. Namely, all mixtures of stabilizer states can be efficiently simulated, but for any number of qubits there also exist efficiently simulable states outside the stabilizer polytope. Further, our simulation method extends to negative quasiprobability distributions, where it provides probability estimation. The simulation cost is then proportional to a robustness measure squared. For all quantum states, this robustness is smaller than or equal to robustness of magic.

Introduction

How to mark the classical-to-quantum boundary is a question that dates back almost to the beginning of quantum theory. Ehrenfest's theorem provides an early insight, and the Einstein-Podolsky-Rosen paradox and Schrödinger's cat are two early puzzles. The advent of quantum computation added a computational angle: When does it become hard to simulate a quantum mechanical computing device on a classical computer? Which quantum mechanical resource do quantum computers harness for a computational speedup?

One instructive computational model is quantum computation with magic states (QCM) [1]. In QCM, both "traditional" indicators of quantumness (developed in the fields of quantum optics and foundations of quantum mechanics) and a computational indicator can be applied. From quantum optics and foundations, the indicators are the negativity of a Wigner function [3], and the breakdown of noncontextual hidden variable models [2]. Computer science is concerned with the breakdown of efficient classical simulation.

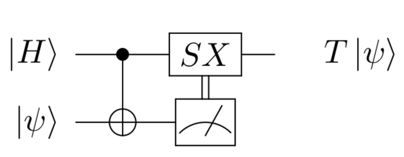
In the particular setting of QCM, an important distinction arises between the cases of even and odd local Hilbert space dimension d. If d is odd, then all three of the above indicators for the classical-to-quantum boundary align [3]. This is a very satisfying situation: the physicist, the philosopher and the computer scientist can have compatible notions of what is "quantum".

In this work, we provide the thus far missing phase space picture for QCM on multi-qubit systems. Central to our discussion is a new quasi-probability function defined for all local Hilbert space dimensions d and all numbers of subsystems n. When applied to odd d, it reproduces the known finite-dimensional adaption of the original Wigner function; but for even d, in particular $d = 2$, it is different. Then, this quasiprobability function requires a phase space of increased size. Even in $d = 2$, the positivity of this quasiprobability is preserved under all Pauli measurements. This property is crucial for the efficient classical simulation of QCM on positively represented states.

QC with magic states (QCM)

In quantum computation by injection of magic states [1],

- The unitaries, measurements and state preparations are non-universal, and are considered "free". The free unitaries are chosen from a subgroup of the Clifford group, and the free measurements are of Pauli observables.
- The computational power resides with the *magic states*, which can be injected into the computation.



Three requirements

A Wigner function used to describe QCM must satisfy three requirements:

(1) Every quantum state ρ has at least one corresponding Wigner function W_ρ.

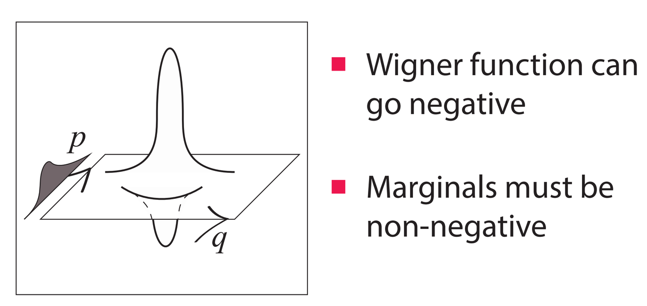
(2) For the measurement of every Pauli observable, the respective outcome probabilities are given by marginals of W.

(3) Under all Pauli measurements, positivity of W is preserved.

Main difficulty with qubits

The main difficulty for establishing a suitable multi-qubit Wigner function is state-independent contextuality w.r.t. Pauli observables, as exhibited by Mermin's square. As a result of this, there is no multi-qubit Wigner function (a) which is positivity-preserving under all Pauli measurements, and (b) whose phase point operators form an operator basis (minimality). We construct a positivity preserving quasiprobability function which is not minimal.

The Discrete Wigner function



$$\boxed{\text{Wigner function negativity is an indicator of quantumess}}$$

The present phase space and Wigner function for n qudits is built from the n-qudit Pauli operators

$$\mathcal{P}_n = \left\{ e^{i\phi(a,b)} \bigotimes_{k=1}^{n} Z^{a_k} X^{b_k} \;:\; a,b \in \mathbb{Z}_d^n \right\}$$

where $X\,|j\rangle = |j+1 \mod d\rangle$, $Z\,|j\rangle = \omega^j\,|j\rangle$, and $\omega = \exp(2\pi i/d)$.

The phase space $\mathcal{V}$ consists of pairs (Ω, λ) where where Ω is a subset of $\mathcal{P}_n$, and $\lambda : \Omega \longrightarrow \{1, \omega, \ldots, \omega^{d-1}\}$ is a function. To satisfy the above three requirements on W, the admissible sets Ω and functions λ must fulfil consistency conditions:

- Ω is closed under inference, i.e., if $P, Q \in \Omega$ and $[P, Q] = 0$ then $PQ \in \Omega$.
- Ω admits a noncontextual value assignment.
- $\lambda : \Omega \to \{1, \omega, \ldots, \omega^{d-1}\}$ is a noncontextual value assignment, i.e. if $P, Q \in \Omega$ commute then $\lambda(PQ) = \lambda(P)\lambda(Q)$.

With (Ω, λ) satisfying these conditions, the phase point operators are

$$A_\Omega^\lambda = \frac{1}{d^n} \sum_{P \in \Omega} \lambda(P) P. \tag{1}$$

The Wigner function, $W_\rho : \mathcal{V} \to \mathbb{R}$, is defined by

$$\rho = \sum_{(\Omega,\lambda)\in\mathcal{V}} W_\rho(\Omega, \lambda) A_\Omega^\lambda. \tag{2}$$

Main Result

Theorem (Efficient classical simulation) If for an initial quantum state ρ it holds that $W_\rho \geq 0$ and furthermore W_ρ can be efficiently sampled from, then the output distribution of all sequences of Pauli measurements, possibly interspersed with Clifford gates, on ρ can be classically efficiently sampled from.

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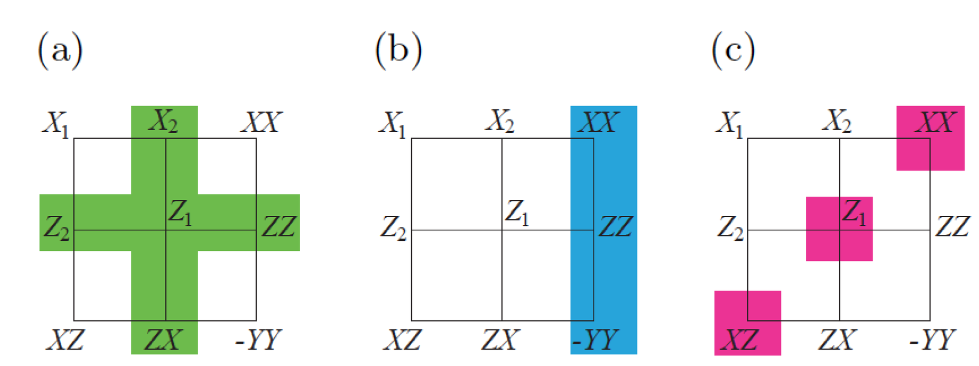
Structure of the phase space

In the case of odd dimensional qudits, $\mathcal{P}_n$ is closed under inference and noncontextual. The only relevant admissible set is $\mathcal{P}_n$ and points in phase space, $\mathcal{V} \cong \mathbb{Z}_d^n \times \mathbb{Z}_d^n$, are labeled by noncontextual value assignments $\lambda : \mathcal{P}_n \to \mathbb{Z}_d$. The phase point operators form an operator basis and the Wigner function is equivalent to the original Wigner function for odd dimensional qudits [3].

In the multi-qubit phase space the admissible sets Ω are an additional varying parameter and so the phase point operators do not form an operator basis, they are overcomplete. A classification of maximal admissible sets Ω for the case of multiple qubits is fully known [4]. Here we provide an example: the classification for the case of two rebits. It is illustrated in the figure below.



Three types of sets Ω are shown for Mermin's square. (a) union of two isotropic subspaces intersecting in one element, (b) isotropic subspace, (c) triple of anti-commuting elements.

The case of $W_\rho < 0$

If the Wigner function W_ρ of the initial magic state ρ has negative entries, then the classical simulation of QCM is no longer efficient. This means, negativity is a necessary computational resource for QCM. In this case, classical simulation is still possible via probability estimation.

The hardness of classical simulation is quantified through a monotone called phase space robustness $\mathfrak{R}$, defined through

$$\mathfrak{R}(\rho) := \min_{W \,|\, \langle \mathcal{A}, W \rangle = \rho} \|W\|_1, \tag{3}$$

with $\langle \mathcal{A}, W \rangle := \sum_{\alpha \in \mathcal{V}} W_\alpha A_\alpha$.

The computational cost N required to estimate the output probabilities scales as

$$N \sim \frac{\mathfrak{R}(\rho_{\text{init}})^2}{\epsilon^2}.$$

Thus, the robustness $\mathfrak{R}(\rho_{\text{init}})$ of the initial state ρ_{init} is the critical parameter determining the classical hardness of probability estimation.

The figure below shows the robustness of three-qubit states

$$|H_3(\phi)\rangle := \left(\frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}} \right)^{\otimes 3},$$

as a function of the angle ϕ (the phase space robustness is in red and the robustness of magic [5] is in blue).