# Simulating quantum computation with magic states: how many "bits" for "it"? 

Michael Zurel ${ }^{1,2}$, Cihan Okay ${ }^{3}$, and Robert Raussendorf ${ }^{1,2}$<br>${ }^{1}$ Department of Physics E Astronomy, University of British Columbia, Vancouver, Canada<br>${ }^{2}$ Stewart Blusson Quantum Matter Institute, University of British Columbia, Vancouver, Canada<br>${ }^{3}$ Department of Mathematics, Bilkent University, Ankara, Turkey


#### Abstract

A recently introduced classical simulation method for universal quantum computation with magic states operates by repeated sampling from probability functions [M. Zurel et al. PRL 260404 (2020)]. This method is closely related to sampling algorithms based on Wigner functions, with the important distinction that Wigner functions can take negative values obstructing the sampling. Indeed, negativity in Wigner functions has been identified as a precondition for a quantum speed-up. However, in the present method of classical simulation, negativity of quasiprobability functions never arises. This model remains probabilistic for all quantum computations. In this paper, we analyze the amount of classical data that the simulation procedure must track. We find that this amount is small. Specifically, for any number $n$ of magic states, the number of bits that describe the quantum system at any given time is $2 n^{2}+O(n)$.


## 1 Introduction

In an article of 1989 [1], John Archibald Wheeler argued that quantum physics required a new perspective on reality based on information theoretic concepts. He wrote: "No element in the description of physics shows itself as closer to primordial than the elementary quantum phenomenon, that is, the elementary deviceintermediated act of posing a yes-no physical question and eliciting an answer or, in brief, the elementary act of observer-participancy. Otherwise stated, every physical quantity, every it, derives its ultimate significance from bits, binary yes-or-no indications, a conclusion we epitomize in the phrase, it from bit."

A prototypical realization of this view on physics has been provided in the description of quantum computation with magic states $(\mathrm{QCM})$ through the $\Lambda$ polytopes $[2]$, in which the quantum phenomena at hand are reproduced, without any deviation or approximation, by repeated sampling processes dependent on and producing bit strings. The " $I t$ " in this case is universal quantum computation, and hence all non-relativistic quantum mechanics in finite-dimensional Hilbert spaces. The "Bits" represent the binary outcomes of Pauli measurements and the labels of the vertices of the $\Lambda$ polytopes in the repeated sampling. There are finitely many such vertices for any number $n$ of magic states.

This description of quantum computation with magic states grew out of the analysis of Wigner function negativity as a precondition for a quantum computational speedup, a research programme that started with Refs. 3 and [4]. Specifically, in Ref. [4] it was established that negativity in Gross' Wigner function [5, 6 is required for a quantum speedup, under the condition that the Hilbert space dimension is odd. Analogous results were subsequently established in even dimension, specifically for rebits [7] and qubits [8]. However, in the end it turned out that once sufficiently general (quasi)probability functions are admitted, there is no need for any negativity at all 2,9 . Universal quantum computation, in the magic state model, can be described by repeated sampling from a generalized phase space whose points are labeled by the vertices of the $\Lambda$ polytopes. This process essentially resembles a random walk, with the complication that the transition function changes from one time step to the next and can depend on the prior sampling history.

A summary of this sampling process is given in Section 2; see in particular Theorem 1 and Algorithm 1 . For now, the gist is displayed in Figure 1. A crucial feature of the $\Lambda$ polytope formalism is that the quantum state $|\Psi(t)\rangle$ of the system (Figure 1a) is replaced by a bit string $b(t)$ of bounded length (Figure 1b). That bit string $b(t)$ is a valid and accurate representation of the quantum system. It should be noted that, in the end, what needs to be reproduced is the quantum mechanical prediction for the joint distribution of measurement outcomes. For it, the statistical distribution of the bit strings $\{b(t), \forall t\}$ matters, not individual values $b(t)$. However, this is the same for the quantum mechanical states $|\Psi(t)\rangle$. They too are conditioned on prior measurement outcomes, hence probabilistic.


Figure 1: Quantum computation with magic states (a), and its simulation based on $\Lambda$ polytopes (b). (a) QCM consists of preparing a quantum register in a magic state $|M\rangle^{\otimes n}$, followed by a sequence of Pauli measurements. This requires a device Prep to deliver the magic states to the quantum register $Q R$, and a classical register CR to store the previous measurement record $\mathbf{s}$, a classical side computation to identify the label $a(t)$ of the Pauli observable measured in step $t$, and a measurement device M to perform the measurements and to output the corresponding results $s(t)$. (b) The overall structure of the classical simulation is the same, but with the components modified. Prep is replaced by a first sampler $\Sigma$ that samples from the phase space distribution of the initial state $|M\rangle^{\otimes n}$. There are two classical registers, CR1 and CR2. The former stores the phase space samples $b(t)$, and the latter the prior measurement record, as in (a). The measurement device M is replaced by a second sampler $\Sigma^{\prime}$ that takes as input a phase space point $b(t)$ and a Pauli label $a(t)$, and outputs a new phase space point $b(t+1)$ as well as a measurement outcome $s(t)$. The same information that in the standard quantum mechanical description is carried by the quantum state $|\Psi(t)\rangle$ is in the $\Lambda$ polytope description carried by the bit string $b(t)$.

The question of interest for the present work is how much classical information the simulation of quantum computations must track, i.e. What is the length of the bit strings $b(t)$ ? For example, if it turned out that those bit strings were very long, say exponentially long in the number $n$ of magic states, this would provide a convenient explanation for the hardness of classical simulation of universal quantum computation using $\Lambda$ polytopes. If the information storage itself is inefficient, so is the processing. However, this is not what we find. We find that the bit strings $b(t)$ are short. Specifically, they are of length $O\left(n^{2}\right)$. Thus, simulation of universal quantum computation based on $\Lambda$ polytopes is a small data problem. The presumed hardness of this simulation must come from the computational hardness of the sampling processes involved, not from moving around large amounts of data.

The length of the bit strings $b(t)$ labeling vertices of $\Lambda_{n}$ is upper-bounded by $\log _{2}\left(\left|\mathcal{V}_{n}\right|\right)$, with $\left|\mathcal{V}_{n}\right|$ the size of the generalized phase space, i.e., the number of vertices of $\Lambda_{n}$. To date, an upper bound and a lower bound are known for this quantity, namely

$$
\frac{n^{2}}{2} \leq \log _{2}\left(\left|\mathcal{V}_{n}\right|\right) \leq 4^{n} n^{2}
$$

The lower bound is by Karanjai, Wallman, and Bartlett [10]. The upper bound comes from the upper bound theorem of polytope theory (11] (see Appendix A for details).

The gap between the bounds is extremely wide, consistent with both efficient and inefficient storage of the bit strings $b(t)$. From numerical results it seems that the number of phase space points is growing very rapidly with $n$; for $n=1$ the number of phase space points is 8 , for $n=2$ it is 22320 , and for $n>2$ we don't know the precise numbers, but the estimate for $n=3$ is already huge.

The following simple insight is crucial for establishing our main result, $|b(t)|=O\left(n^{2}\right)$ : in the QCM model, for any fixed value $n$, all quantum computations start in the same magic state $|M\rangle^{\otimes n}$. Therefore, the question of interest for classical simulation of QCM using $\Lambda$ polytopes is not "What is the size of the phase space $\mathcal{V}_{n}$ ?", but rather "What is the size of the region of $\mathcal{V}_{n}$ that can be reached from the initial magic state $|M\rangle^{\otimes n}$ ?". This size can be computed, and it turns out to be small. It implies a new upper bound for the length of the bit strings $b(t)$ that is within a factor of four of the matching lower bound [10]. This is the content of our main result, Theorem 2, presented in Section 3 below.

## 2 Preliminaries

In Ref. [2], a hidden variable model (HVM) is defined for quantum computation with magic states (QCM) [12]a universal model of quantum computation in which computation proceeds through a sequence of Clifford gates and Pauli measurements on an initially prepared "magic state".

The measurements come from the $n$-qubit Pauli group $\mathcal{P}_{n}$, the group generated by the Pauli operators $X, Y, Z$ acting on $n$ qubits. Modding out overall phases we have $\mathcal{P}_{n} / \mathcal{Z}\left(\mathcal{P}_{n}\right) \cong \mathbb{Z}_{2}^{2 n}$, and WLOG we can fix a phase convention for the Pauli operators to be

$$
T_{a}=i^{-\left\langle a_{z} \mid a_{x}\right\rangle} \bigotimes_{k=1}^{n} Z^{a_{z}[k]} X^{a_{x}[k]} ; \quad \forall a=\left(a_{z}, a_{x}\right) \in \mathbb{Z}_{2}^{n} \times \mathbb{Z}_{2}^{n}=: E_{n}
$$

where the inner product $\left\langle a_{z} \mid a_{x}\right\rangle$ is computed modulo 4 . The projector corresponding to a Pauli measurement $a \in E_{n}$ yielding outcome $s \in \mathbb{Z}_{2}$ is denoted $\Pi_{a}^{s}:=\left(I+(-1)^{s} T_{a}\right) / 2$. The gates of the model are chosen from the Clifford group, which is the normalizer of the Pauli group in the unitary group up to overall phases: $\mathcal{C} \ell_{n}=\mathcal{N}\left(\mathcal{P}_{n}\right) / U(1)$. They are defined by the property that they map Pauli operators to Pauli operators under conjugation.

We denote by $\operatorname{Herm}(\mathcal{H})$ the space of Hermitian operators on Hilbert space $\mathcal{H}$, and unless otherwise specified, $\mathcal{H}$ (or $\mathcal{H}_{n}$ ) is the $n$-qubit Hilbert space $\left(\mathbb{C}^{2}\right)^{\otimes n}$. $\operatorname{Herm}_{1}(\mathcal{H})$ is the affine subspace of $\operatorname{Herm}(\mathcal{H})$ consisting of operators with unit trace and $\operatorname{Herm}_{1}^{\succeq 0}(\mathcal{H})$ is the subset of $\operatorname{Herm}_{1}(\mathcal{H})$ consisting of positive semidefinite operators. $\operatorname{Herm}_{1}^{\succeq 0}(\mathcal{H})$ contains the density operators representing physical quantum states.

The state space of the hidden variable model of Ref. [2] is based on the $\Lambda$ polytopes. Denoting the set of pure $n$-qubit stabilizer states by $\mathcal{S}_{n}$, the $\Lambda$ polytope for $n$ qubits is defined as

$$
\begin{equation*}
\left.\Lambda_{n}=\left\{X \in \operatorname{Herm}_{1}\left(\mathcal{H}_{n}\right)|\operatorname{Tr}(|\sigma\rangle\langle\sigma| X) \geq 0 \forall| \sigma\right\rangle \in \mathcal{S}_{n}\right\} \tag{1}
\end{equation*}
$$

For a fixed number $n \in \mathbb{N}$ of qubits, $\Lambda_{n}$ is a bounded polytope with a finite number of vertices [9]. We denote the vertices of $\Lambda_{n}$ by $\left\{A_{\alpha} \mid \alpha \in \mathcal{V}_{n}\right\}$ where $\mathcal{V}_{n}$ is an index set for the vertices. The hidden variable model is defined by the following theorem.

Theorem 1 (Ref. [2]; Theorem 1) For any number of qubits $n \in \mathbb{N}$,

1. Any $n$-qubit quantum state $\rho \in \operatorname{Herm}_{1}^{\succeq 0}\left(\mathcal{H}_{n}\right)$ can be decomposed as

$$
\begin{equation*}
\rho=\sum_{\alpha \in \mathcal{V}_{n}} p_{\rho}(\alpha) A_{\alpha} \tag{2}
\end{equation*}
$$

with $p_{\rho}(\alpha) \geq 0$ for all $\alpha \in \mathcal{V}_{n}$, and $\sum_{\alpha} p_{\rho}(\alpha)=1$. I.e. any $n$-qubit quantum state $\rho$ can be represented by a probability distribution $p_{\rho}$ over $\mathcal{V}_{n}$.
2. For any $A_{\alpha}, \alpha \in \mathcal{V}_{n}$, and any Clifford gate $g \in \mathcal{C} \ell_{n}, g A_{\alpha} g^{\dagger}$ is a vertex of $\Lambda_{n}$. This defines an action of the Clifford group on $\mathcal{V}_{n}$ as $g A_{\alpha} g^{\dagger}=: A_{g \cdot \alpha}$ where $g \cdot \alpha \in \mathcal{V}_{n}$.
3. For any $A_{\alpha}, \alpha \in \mathcal{V}_{n}$, and any Pauli projector $\Pi_{a}^{s}$, we have

$$
\begin{equation*}
\Pi_{a}^{s} A_{\alpha} \Pi_{a}^{s}=\sum_{\beta \in \mathcal{V}_{n}} q_{\alpha, a}(\beta, s) A_{\beta} \tag{3}
\end{equation*}
$$

with $q_{\alpha, a}(\beta, s) \geq 0$ for all $\beta \in \mathcal{V}_{n}$ and $s \in \mathbb{Z}_{2}$, and $\sum_{\beta, s} q_{\alpha, a}(\beta, s)=1$. I.e. Pauli measurements are represented by a stochastic map from (phase-space-point, measurement) pairs $(\alpha, a) \in \mathcal{V}_{n} \times E_{n}$ to (phase-space-point, measurement outcome) pairs $(\beta, s) \in \mathcal{V}_{n} \times \mathbb{Z}_{2}$.

A classical simulation algorithm for QCM based on sampling from the defining probability distributions of this HVM is given in Algorithm 1 . The algorithm returns samples from the distribution of measurement outcomes for the quantum circuit being simulated which agree with the predictions of quantum theory [2].

```
Algorithm 1 Classical simulation of a single run of a magic state quantum circuit with input state \(\rho\)
    sample \(\alpha \in \mathcal{V}_{n}\) according to \(p_{\rho}: \mathcal{V}_{n} \rightarrow \mathbb{R}_{\geq 0}\)
    propagate \(\alpha\) through the circuit
    while the end of the circuit has not been reached do
        if a Clifford gate \(g \in \mathcal{C} \ell_{n}\) is encountered then
            update the phase space point according to \(\alpha \mapsto g \cdot \alpha\)
        end if
        if a Pauli measurement \(a \in E_{n}\) is encountered then
            sample \((\beta, s) \in \mathcal{V}_{n} \times \mathbb{Z}_{2}\) according to \(q_{\alpha, a}\)
            return \(s \in \mathbb{Z}_{2}\) as the outcome of the measurement
            update the phase space point according to \(\alpha \mapsto \beta\)
        end if
    end while
```


## 3 Main result

In this section we present our main result. First, in section 3.1 we give the result for a simplified version of the computational model wherein we only allow Pauli measurements, no Clifford gates, and in particular, we allow only sequences of independent and commuting Pauli measurements. This simplified model is still universal for quantum computation [13, 14]. Then, in Section 3.2 we give a more general statement of the main result where we allow computations consisting of arbitary sequences of Clifford gates and Pauli measurements.

### 3.1 Simplified case

Although the most general quantum computation in QCM could consist of any sequence of Clifford gates and Pauli measurements performed on an arbitrary input state, we can make several assumptions simplifying the computational model while preserving the property of quantum computational universality.

First, in QCM, we usually assume that every computation starts from a fixed magic input state. For example, using the standard magic state circuit gadget [15, Figure 10.25], any Clifford $+T$ circuit with $n$ $T$-gates acting on $m$ qubits can be turned into a QCM circuit (Clifford gates and Pauli measurements only) on $n+m$ qubits acting on a state of the form $|0\rangle^{\otimes m} \otimes|H\rangle^{\otimes n}$ where $|H\rangle=(|0\rangle+\exp (i \pi / 4)|1\rangle) / \sqrt{2}$. In general, it suffices to consider input states of the form $|0\rangle^{\otimes m} \otimes|M\rangle^{n}$ where $|M\rangle$ is a fixed magic state 16 .

Second, we can do away with the Clifford gates altogether 7,13 . To see this, note that the Clifford gates can always be propagated forward in time through the circuit, conjugating the Pauli measurements into other Pauli measurements. Once they are propagated past the final measurement in the circuit they can be removed since they no longer affect the statistics of the measurement outcomes.

We can also do away with the stabilizer part of the input. The Pauli circuit on the $m+n$-qubit input state $|0\rangle^{\otimes m} \otimes|M\rangle^{\otimes n}$ can be simulated by another circuit consisting of an adaptive sequence of Pauli measurements acting only on the $n$-qubit magic part of the input [14, 17]. The stabilizer part is handled by extra classical processing.

Finally, it suffices to consider only sequences of commuting Pauli measurements up to length $n$ 14. This is because when a Pauli measurement is encountered which anticommutes with a measurement that was previously performed, the outcome of the measurement will be uniformly random, and the update of the state after the measurement can be implemented by a Clifford gate. Therefore, the anticommuting measurement can always for replaced by a coin flip to determine the measurement outcome and a Clifford gate to implement the state update which can then be propagated past the future measurements. The longest sequences of independent pair-wise commmuting Pauli measurements on $n$ qubits have length $n$.

To summarize, for universal quantum computation, it suffices to consider adaptive sequences of pair-wise commuting Pauli measurements of length $n$ acting on a fixed magic state of the form $|M\rangle^{\otimes n}$. A simplified version of the classical simulation algorithm above for circuits of this form is given by Algorithm 2, For a complete description of the above circuit simplifications, along with a method for compiling a given QCM circuit into an adaptive Pauli circuit, see Ref. (14.

```
Algorithm 2 Classical simulation of a single run of a Pauli-based quantum circuit with input state \(\rho\)
    sample \(\alpha_{0} \in \mathcal{V}_{n}\) according to \(p_{\rho}: \mathcal{V}_{n} \rightarrow \mathbb{R}_{\geq 0}\)
    for all \(a_{t}, t \in\{1,2, \ldots, n\}\) do
        sample \(\left(\alpha_{t}, s_{t}\right) \in \mathcal{V}_{n} \times \mathbb{Z}_{2}\) according to \(q_{\alpha_{t-1}, a_{t}}\)
        return \(s_{t} \in \mathbb{Z}_{2}\) as the outcome of measurement \(a_{t}\)
        update the phase space point according to \(\alpha_{t-1} \mapsto \alpha_{t}\)
    end for
```

We can now state our main result.
Theorem 2 (Main result) Any quantum computation consisting of a sequence of $n$ independent, pair-wise commuting Pauli measurements on a fixed magic state $|M\rangle^{\otimes n}$ can be simulated by Algorithm 2 using a memory of $2 n^{2}+3 n$ bits to specify the phase space points reached.

The proof of this theorem relies on the following result from convex geometry.
Lemma 1 (Carathéodory's theorem) If a point $x$ of $\mathbb{R}^{D}$ lies in the convex hull of a set $V$, then $x$ can be written as the convex combination of at most $D+1$ points in $V$.

For a proof of this lemma see for example Ref. [18, §1.6].
Proof of Theorem 2. Since the generalized phase space point operators $\left\{A_{\alpha} \mid \alpha \in \mathcal{V}_{n}\right\}$ are not a basis for $\operatorname{Herm}_{1}\left(\mathcal{H}_{n}\right)$, they are overcomplete, the distributions $p_{\rho}$ that represent states in the model of Theorem 1 . and the distributions $q_{\alpha, a}$ that represent Pauli measurements are not unique. Since $\Lambda_{n}$ lives in Herm $\operatorname{He}_{1}\left(\mathcal{H}_{n}\right)$, a real affine space of dimension $4^{n}-1$, by Lemma 1 there exist choices for the distribution $p_{\rho}$ such that $\left|\operatorname{supp}\left(p_{\rho}\right)\right| \leq 4^{n}$. Similarly, for each $s \in \mathbb{Z}_{2}$, there exist choices for $q_{\alpha, a}(-, s)$ such that $\left|\operatorname{supp}\left(q_{\alpha, a}(-, s)\right)\right| \leq 4^{n}$. To start we fix a canonical choice of the probability distributions $p_{\rho}$ and $q_{\alpha, a}$ satisfying these properties. With this canonical choice for the distribution representing the input state of the circuit, specifying a sample from this distribution requires no more than $\log _{2}\left(4^{n}\right)=2 n$ bits.

There are $4^{n}-1$ nontrivial $n$-qubit Pauli measurements, therefore, specifying each measurement requires no more than $2 n$ bits. For the $t^{t h}$ measurement $a_{t}$, the distribution $q_{\alpha_{t-1}, a_{t}}$ is uniquely specified by the sampling history consisting of states $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{t-1}$, measurements $a_{1}, a_{2}, \ldots, a_{t-1}$, and measurement outcomes $s_{1}, s_{2}, \ldots, s_{t-1}$. Once the distribution is fixed, with the canonical choice above, specifying a sample from this distribution requires no more than $2 n+1$ bits ( 1 bit for $s_{t}$ and $2 n$ bits for $\alpha_{t}$ ).

Since the length of the measurement sequence is no more than $n$, the number of classical bits required to specify the complete sampling history is no more than

$$
\underbrace{2 n}_{\alpha_{0}}+\sum_{t=1}^{n}[\underbrace{2 n}_{a_{t}}+\underbrace{1}_{s_{t}}+\underbrace{2 n}_{\alpha_{t}}]=4 n^{2}+3 n
$$

This initial bound can be improved in a number of ways. First, for the purpose of simulation, we don't need to store $\alpha_{n}$ since there are no more measurements. This immediately removes $2 n$ bits. Second, after the $t^{t h}$ measurement, the value of any measurement in the span of $a_{1}, a_{2}, \ldots, a_{t}$ is already determined. Therefore, if the $t+1^{t h}$ measurement $a_{t+1}$ is to be independent and commute with the previous measurements, it is chosen from $\left(\mathbb{Z}_{2}^{2 n} \cap\left\{a_{1}, a_{2}, \ldots, a_{t}\right\}^{\perp}\right) / \operatorname{span}\left(a_{1}, a_{2}, \ldots, a_{t}\right) \cong \mathbb{Z}_{2}^{2(n-t)}$. Specifying a measurement chosen from this set requires only $2(n-t)$ bits, not the full $2 n$ bits.

Finally, we can perform another simplification which reduces the number of qubits by 1 after each measurement. After a measurement of $a \in E_{n}$ giving outcome $s \in \mathbb{Z}_{2}$, the relevant state space is projected to $\Pi_{a}^{s} \Lambda_{n} \Pi_{a}^{s}$. This is contained in a $4^{n-1}-1$ dimensional subspace of $\operatorname{Herm}_{1}\left(\mathcal{H}_{n}\right)$. There exists a Clifford gate $g \in \mathcal{C} \ell_{n}$ such that $g \Pi_{z_{n}}^{0} g^{\dagger}=\Pi_{a}^{s}$, and $\Pi_{a}^{s} \Lambda_{n} \Pi_{a}^{s}=g \Pi_{z_{n}}^{0} \Lambda_{n} \Pi_{z_{n}}^{0} g^{\dagger}=g\left(\Lambda_{n-1} \otimes|0\rangle\langle 0|\right) g^{\dagger}$. Therefore, after the measurement the Clifford gate $g$ can be propagated out yielding a Pauli measurement circuit on an input state with a stabilizer state tensor factor. Then the above mentioned circuit simplification [14, 17] can be used to remove the stabilizer part of the input. This reduction can be performed after each measurement. This is similar to the idea behind the reduced classical simulation of Theorem 3 of Ref. [17. With this dimension

(d)

Figure 2: Three descriptions of a quantum computation. (a) A circuit level description. Pauli measurements $a_{1}, a_{2}, \ldots$ are performed on the input state $\rho$ yielding measurement outcomes $s_{1}, s_{2}, \ldots$ (b) The representation of this computation in the model of Theorem 1. The input state is represented by the probability distribution $p_{\rho}$, each measurement $a_{t}$ is represented by a probability distribution $q_{\alpha_{t-1}, a_{t}}$. (c) A single run of the simulation algorithm based on the probabilistic representation of the computation. We start by sampling from $p_{\rho}$ to obtain $\alpha_{0}$. For each measurement $a_{i}$ we sample from $q_{\alpha_{t-1}, a_{t}}$ to obtain measurement outcome $s_{t}$ and updated state $\alpha_{t}$. (d) An upper bound on the number of classical bits required to store each piece of the sampling history of (c) according to Theorem 2 .
reduction after each measurement, to specify the sample $\left(\alpha_{t}, s_{t}\right)$ requires no more than $2(n-t)+1$ bits. Note that this dimension reduction implicitly implements both of the reductions of the previous paragraph.

With these reductions, the complete measurement history can be specified with no more than

$$
\underbrace{2 n}_{\alpha_{0}}+\sum_{t=1}^{n}[\underbrace{2(n-t+1)}_{a_{t}}+\underbrace{1}_{s_{t}}+\underbrace{2(n-t)}_{\alpha_{t}}]=2 n^{2}+3 n
$$

classical bits, which is the claimed bound.
A similar idea as the one used in the proof of Theorem 2 of explicitly tracking the measurement history of a quantum circuit has previously been used to define a contextual hidden variable model for the stabilizer subtheory 19.

### 3.2 General case

Theorem 2 used a simplified version of QCM, where, without loss in computational power, all Clifford gates have been eliminated, and the measurement sequence is shrunk to at most $n$ commuting Pauli measurements. In the remainder of this section we demonstrate that the bound on the size of the reachable phase space region does not increase much if we do not make these simplifications; i.e., if we admit arbitrarily long sequences of (potentially non-commuting) Pauli measurements, and Clifford gates between them. Specifically, we have the following additional result.

Corollary 1 Any quantum computation consisting of an arbitrarily long sequence of Pauli measurements and Clifford unitaries, applied to a fixed magic state $|M\rangle^{\otimes n}$, can be simulated using a memory of $4 n^{2}+6 n$ bits to specify the reachable phase space points.

Comparing with Theorem 2 we find that the memory requirement merely doubles; in particular, the quadratic scaling with the number $n$ of magic states remains unchanged.

Proof of Corollary 1. Every vertex of $\Lambda_{n}$ is in exactly one orbit of vertices with respect to the action of the Clifford group. Therefore, the number of Clifford orbits travelled by Algorithm 2 is smaller than or at most equal to the number of vertices travelled. A consequence of Theorem 2 thus is that the number of equivalence classes travelled by Algorithm 2 is no more than $2^{2 n^{2}+3 n}$.

At every given point in the classical simulation, the single vertex $A_{\nu}$ under consideration by the simulation algorithm has a product structure, $A_{\nu}=g\left(\tilde{A}_{\nu} \otimes|S\rangle\langle S|\right) g^{\dagger}$, where $|S\rangle\langle S|$ is the projector onto an $m$-qubit stabilizer state with stabilizer group $S, \tilde{A}_{\nu}$ is a vertex of $\Lambda_{n-m}$, and $g$ is a Clifford unitary 17 .

When switching from Algorithm 2 specialized to the simplified computational model back to the general Algorithm 1, the following additional situations may occur at any given step: (i) the next operation is a Clifford unitary, (ii) the next operation is the measurement of a Pauli observable $T_{a}, a \in E_{n}$, such that $\pm T_{a} \in g S g^{\dagger}$, and (iii) the next operation is the measurement of a Pauli observable $T_{a}$ that does not commute with all elements of $g S g^{\dagger}$.

In case (i), the Clifford orbit $\left[A_{\nu}\right]$ of $A_{\nu}$ doesn't change. In case (ii), the measurement outcome deterministically follows from the stabilizer, and no update of $A_{\nu}$ occurs at all. Hence, no update of $\left[A_{\nu}\right]$ occurs either. In case (iii), the measurement outcome is uniformly random, and for either outcome the ensuing transformation can be replaced by a Clifford unitary. Thus we are back to case (i)—no update of $\left[A_{\nu}\right]$ occurs. The conclusion is that the number of Clifford orbits of vertices reachable by the more general Algorithm 1 equals the number of orbits reachable by the specialized Algorithm 2 applying to the canonical form of QCMs.

Now, the size of every Clifford orbit of vertices of $\Lambda_{n}$ is upper-bounded by the size $\left|\mathcal{C} \ell_{n}\right|$ of the n-qubit Clifford group. Therefore, the number of vertices reachable by Algorithm 11 is bounded by $2^{2 n^{2}+3 n} \times\left|\mathcal{C} \ell_{n}\right|$. The number of bits required to specify an $n$-qubit Clifford gate is no more than $2 n^{2}+3 n$ (namely, $2 n \times(2 n+1$ ) bits to specify the conjugation relations of a generating set of Pauli operators including the sign, minus $2 n^{2}-n$ constraints for preserving the Pauli commutation relations). Hence, the number of bits needed to specify any vertex reached in the simulation according to Algorithm 1 is no more than $4 n^{2}+6 n$, as claimed.

## 4 Discussion

To summarize, in this work we have shown that the classical simulation of universal quantum computation using the $\Lambda$-polytopes $[2]$ is a small data problem. Specifically, it is shown that, with respect to the model of quantum computation with magic states, the number of bits that represent the quantum system at any stage of the simulation is quadratic in the number $n$ of the magic states. Classical simulation of quantum computation is (presumably) still hard, but in the present case this hardness does not stem from shuffling around lots of data but instead from complicated operations on little data.

For illustration, we compare the above classical simulation method to two others, one very different, one rather similar. The first method is the straightforward simulation of a quantum system obtained by choosing a specific basis of the Hilbert space at hand, mapping operators to matrices and states to vectors. The state of the system at any moment in time is now described by exponentially many complex-valued amplitudes. The bit equivalent of a complex number is a matter of numeric precision, but in any case the amount of data to be processed is large. The state update is conceptually simple, and the computational hardness derives from the size of the objects involved. This simulation method and that of Ref. [2] thus represent opposite ends of the spectrum.

The second simulation method we compare to is that of sampling from Wigner functions [4], applicable to odd Hilbert space dimension. Here, the overall structure of the simulation is the same as in Ref. 2], i.e., repeated sampling from a phase space. There are two important differences, however. In Ref. [4], (i) the sampling is computationally efficient whenever it applies, but (ii) the sampling procedure does not apply to all initial magic states. Specifically, it only applies when the Wigner function [5, 6] of the initial state is positive. Indeed, this is why negativity of the Wigner function is a precondition for computational speedup. In Ref. [2], (i) the sampling is not guaranteed to be computationally efficient, but (ii) it applies to all possible initial states.

We highlight two further aspects of the simulation method [2]:

- In the classical simulation of quantum computation using $\Lambda$-polytopes, the description of the system's state by $2 n^{2}+3 n$ bits does not invoke any approximation. The distributions of measurement outcomes sampled from are the exact quantum-mechanical ones. Thus, the data representing the system is genuinely discret ${ }^{1}$, and for this reason we regard 2 as a realization of Wheeler's "it from bit" proposal.

[^0]- Theorem 2 can be generalized to the odd-prime-dimensional qudit case [9]. In this case, the precompilation step reducing the computation to sequences of commuting Pauli measurements is also possible 20, but again not necessary.

To conclude, we observe that the $\Lambda$-polytopes are only beginning to be explored, and they may hold many more surprises.

Acknowledgements. MZ and RR thank William G. Unruh for insightful discussions. MZ and RR are funded by NSERC, in part through the Canada First Research Excellence Fund, Quantum Materials and Future Technologies Program. CO is supported by the US Air Force Office of Scientific Research under the award number FA9550-21-1-0002. This work was supported by the Horizon Europe project FoQaCiA, GA no. 101070558.

## References

[1] John Archibald Wheeler. "Information, Physics, Quantum: The Search for Links". In: Proceedings of the 3rd International Symposium on Foundations of Quantum Mechanics in the Light of New Technology. Ed. by Hiroshi Ezawa, Shun Ichi Kobayashi, and Yoshimasa Murayama. (Also published in Ref. 21]). Physical Society of Japan, 1989, pp. 354-368.
[2] Michael Zurel, Cihan Okay, and Robert Raussendorf. "Hidden Variable Model for Universal Quantum Computation with Magic States on Qubits". In: Physical Review Letters 125.26 (2020), p. 260404. Doi: 10.1103/PhysRevLett.125.260404. arXiv: 2004.01992.
[3] Ernesto F. Galvão. "Discrete Wigner functions and quantum computational speedup". In: Physical Review A 71.4 (2005), p. 042302. DOI: 10.1103/PhysRevA.71.042302, arXiv: quant-ph/0405070.
[4] Victor Veitch, Christopher Ferrie, David Gross, and Joseph Emerson. "Negative quasi-probability as a resource for quantum computation". In: New Journal of Physics 14.11 (2012), p. 113011. DOI: 10 . 1088/1367-2630/14/11/113011. arXiv: 1201.1256.
[5] David Gross. "Hudson's theorem for finite-dimensional quantum systems". In: Journal of Mathematical Physics 47.12 (2006), p. 122107. DOI: 10.1063/1.2393152, arXiv: quant-ph/0602001.
[6] David Gross. "Computational power of quantum many-body states and some results on discrete phase spaces". PhD Thesis. Institute for Mathematical Sciences, Imperial College London, 2008. url: https: //www.thp.uni-koeln.de/gross/files/diss.pdf.
[7] Nicolas Delfosse, Philippe Allard Guerin, Jacob Bian, and Robert Raussendorf. "Wigner Function Negativity and Contextuality in Quantum Computation on Rebits". In: Physical Review X 5.2 (2015), p. 021003. DOI: 10.1103/PhysRevX.5.021003. arXiv: 1409.5170
[8] Robert Raussendorf, Juani Bermejo-Vega, Emily Tyhurst, Cihan Okay, and Michael Zurel. "Phase-space-simulation method for quantum computation with magic states on qubits". In: Physical Review A 101.1 (2020), p. 012350. DOI: 10.1103/PhysRevA.101.012350. arXiv: 1905.05374 .
[9] Michael Zurel, Cihan Okay, Robert Raussendorf, and Arne Heimendahl. "Hidden variable model for quantum computation with magic states on qudits of any dimension". 2021. arXiv: 2110.12318.
[10] Angela Karanjai, Joel J. Wallman, and Stephen D. Bartlett. "Contextuality bounds the efficiency of classical simulation of quantum processes". 2018. arXiv: 1802.07744 .
[11] Peter McMullen. "The maximum numbers of faces of a convex polytope". In: Mathematika 17.2 (1970), pp. 179-184. DOI: 10.1112/S0025579300002850.
[12] Sergey Bravyi and Alexei Kitaev. "Universal quantum computation with ideal Clifford gates and noisy ancillas". In: Physical Review A 71.2 (2005), p. 022316. DOI: 10.1103/PhysRevA.71.022316. arXiv: quant-ph/0403025.
[13] Sergey Bravyi, Graeme Smith, and John A Smolin. "Trading classical and quantum computational resources". In: Physical Review Letters 6.2 (2016), p. 021043. DOI: 10.1103/PhysRevX. 6.021043. arXiv: 1506.01396.
[14] Filipa C.R. Peres and Ernesto Galvao. "Quantum circuit compilation and hybrid computation using Pauli-based computation". 2021. arXiv: 2203.01789.
[15] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press, 2010. DOI: 10.1017/CB09780511976667.
[16] Ben W. Reichardt. "Quantum universality by state distillation". In: Quantum Information and Computation 9.11 (2009), pp. 1030-1052. DOI: https://doi . org/10.26421/QIC9.11-12-7. arXiv: quant-ph/0608085.
[17] Cihan Okay, Michael Zurel, and Robert Raussendorf. "On the extremal points of the $\Lambda$-polytopes and classical simulation of quantum computation with magic states". In: Quantum Information 8 Computation 21.13\&14 (2021). DOI: 10.26421/QIC21.13-14-2. arXiv: 2104.05822.
[18] Günter M. Ziegler. Lectures on Polytopes. Springer Science+Business Media, 1995. Dor: 10.1007/978-1-4613-8431-1.
[19] Angela Karanjai. "Statistical Modelling of Quantum Data". PhD Thesis. School of Physics, University of Sydney, 2019. URL: https://hdl.handle.net/2123/22134.
[20] Filipa C.R. Peres. "The Pauli-based model of quantum computation with higher dimensional systems". 2023. arXiv: 2302.13702.
[21] John Archibald Wheeler. "Information, Physics, Quantum: The Search for Links". In: Complexity, Entropy and the Physics of Information. Ed. by Wokciech H. Zurek. 1st Edition. The proceedings of the 1988 Workshop on Complexity, Entropy, and the Physics of Information, held May-June, 1989 in Santa Fe, New Mexico. CRC Press, LLC, 1990, pp. 3-28. DoI: 10.1201/9780429502880.
[22] Branko Grünbaum. Convex Polytopes. Springer New York, NY, 2003. DOI: 10. 1007/978-1-4613-0019-9.

## A Upper bound on the number of vertices of $\Lambda$

Lemma 2 For any number of qubits $n \in \mathbb{N}$, the number of vertices $\left|\mathcal{V}_{n}\right|$ of $\Lambda_{n}$ satisfies

$$
\log _{2}\left(\left|\mathcal{V}_{n}\right|\right) \leq 1+4^{n-1}\left[n^{2}-n+2 \log _{2}(6 e)\right]
$$

The proof relies on the following lemma bounding the number of stabilizer states for $n$ qubits.
Lemma 3 For any number of qubits $n \in \mathbb{N}$, the number of pure $n$-qubit stabilizer states is $\left|\mathcal{S}_{n}\right| \leq 3 \cdot 2^{\left(n^{2}+3 n\right) / 2}$.
Proof of Lemma 3. This can be shown by induction. For the base case, we have $\left|S_{1}\right|=6 \leq 3 \cdot 2^{\left(1^{2}+3\right) / 2}=12$. Now, assume $\left|\mathcal{S}_{n}\right| \leq 3 \cdot 2^{\left(n^{2}+3 n\right) / 2}$ for some $n \in \mathbb{N}$. Then

$$
\begin{aligned}
\left|\mathcal{S}_{n+1}\right| & =2^{n+1} \prod_{k=1}^{n+1} 2^{k}+1 \\
& =2\left(2^{n+1}+1\right) \cdot\left|\mathcal{S}_{n}\right| \\
& \leq 2\left(2^{n+1}+1\right) \cdot 3 \cdot 2^{\left(n^{2}+3 n\right)} \\
& \leq 3 \cdot 2^{\left[(n+1)^{2}+3(n+1)\right] / 2}
\end{aligned}
$$

Therefore, by induction, $\left|\mathcal{S}_{n}\right| \leq 3 \cdot 2^{\left(n^{2}+3 n\right) / 2}$ for all $n \in \mathbb{N}$.
Proof of Lemma 2, According to the upper bound theorem of polytope theory [11] (also see Ref. [18, $\S 8.4]$ ), the number of facets of a $D$-dimensional polytope with $v$ vertices is bounded by the number of facets of the $D$-dimensional cyclic polytope with $v$ vertices, denoted $C(v, D)$. By duality, the number of vertices of the $\Lambda$ polytope on $n$ qubits is bounded by the number of vertices of the polar dual of the $4^{n}-1$-dimensional cyclic polytope with $\left|\mathcal{S}_{n}\right|$ vertices.

The number of vertices of the dual of $C(v, D)$ is $22, \S 4.7]$,

$$
f_{d-1}(C(v, D))= \begin{cases}\frac{v}{v-m}\binom{v-m}{m} & \text { for even } D=2 m \\ 2\binom{v-m-1}{m} & \text { for odd } D=2 m+1\end{cases}
$$

In the case of $\Lambda_{n}, D=4^{n}-1$, i.e. $D$ is odd and $m=2^{2 n-1}-1$, and the number of facets is $\left|\mathcal{S}_{n}\right|$. Therefore,

$$
\begin{aligned}
\left|\mathcal{V}_{n}\right| & \leq f_{4^{n}-2}\left(C\left(\left|\mathcal{S}_{n}\right|, 4^{n}-1\right)\right. \\
& =2\binom{\left|\mathcal{S}_{n}\right|-2^{2 n-1}}{2^{2 n-1}-1} \\
& \leq 2\binom{\left|\mathcal{S}_{n}\right|-2^{2 n-1}}{2^{2 n-1}} \\
& \leq 2\left(\frac{e\left(\left|\mathcal{S}_{n}\right|-2^{2 n-1}\right)}{2^{2 n-1}}\right)^{2^{2 n-1}}
\end{aligned}
$$

In the last line we use a standard upper bound for the binomial coefficient $\binom{n}{k} \leq\left(\frac{e n}{k}\right)^{k}$. Then

$$
\begin{aligned}
\log _{2}\left(\left|\mathcal{V}_{n}\right|\right) & \leq 1+2^{2 n-1}\left[\log _{2}\left(\mid \mathcal{S}_{n}-2^{2 n-1}\right)+\log _{2}(e)-2 n+1\right] \\
& \leq 1+2^{2 n-1}\left[\log _{2}\left(3 \cdot 2^{\left(n^{2}+3 n\right) / 2}\right)-2 n+1+\log _{2}(e)\right] \\
& \leq 1+2^{2 n-1}\left[\frac{n^{2}+3 n}{2}-2 n+1+\log _{2}(3 e)\right] \\
& =1+4^{n-1}\left[n^{2}-n+2 \log _{2}(6 e)\right] .
\end{aligned}
$$

This proves the upper bound.
We also have the following somewhat simpler bound.
Corollary 2 For any number of qubits $n \in \mathbb{N}, \log _{2}\left(\left|\mathcal{V}_{n}\right|\right) \leq 4^{n} n^{2}$.
Proof of Corollary 2. For $n=1$ and $n=2$, we can enumerate the vertices of $\Lambda_{n}$ and we find that the numbers of vertices are 8 and 22320 respectively. These both satisfy the bound. Further, for $n \geq 2$, $1+4^{n-1}\left[n^{2}-n+2 \log _{2}(6 e)\right] \leq 4^{n} n^{2}$ and so the remaining cases follow immediately from Lemma 2 ,


[^0]:    ${ }^{1}$ Any continuous parameter dependence is relegated to the sampling probabilities in the first sampling step.

